

Theory of Computation
CS 378
Fall 2019

Problem Set 1

Due **AT THE BEGINNING OF CLASS** Wednesday, September 19

- Please read Chapter 0 and Chapter 1, pages 31–47, of Sipser.
- *A general note:* When writing up your homework, please explain your arguments clearly and write neatly. Graders may not award credit to incomplete or illegible solutions. Clear communication *is* the point, on every assignment.
- *A general note:* Neatly written (or typeset) solutions, with enough blank space on the page to allow graders to write comments before returning the papers, are greatly appreciated!
- You may not use any material other than your Sipser textbook (and lecture notes) for exercise 1 below.

1. Sipser, Exercise 0.7. For simplicity, your relations should use the domain $A \times A$ where $A = \{1, 2, 3\}$. (*Note:* You are being asked to give an actual relation for each part. Write your relations as sets of ordered pairs as on page 9 of Sipser.)
2. One formal technique for proving that two sets X and Y are equal is to prove that $X \subseteq Y$ and that $Y \subseteq X$. To prove that $X \subseteq Y$, we must prove that for any x , if $x \in X$, then $x \in Y$. Similarly, to prove that $Y \subseteq X$, we must prove that for any y , on the assumption that $y \in Y$, it is also true that $y \in X$. For example, in Theorem 0.20, Sipser formally proves that $\overline{A \cup B} = \overline{A} \cap \overline{B}$ (although it could perhaps be argued that a few small details are missing from that proof!).

For this exercise, your task is to give a formal proof that $\overline{A \cap B} = \overline{A} \cup \overline{B}$. (This and the result of Theorem 0.20 are collectively known as *DeMorgan's Laws*.)

3. The *cardinality* of a finite set A is the number of elements in A . For example, the cardinality of $\{b, l, a, h\}$ is 4 and the cardinality of $\{b, \{l, a, h\}\}$ is 2 (do you see why?). The cardinality of a set A is denoted $|A|$.

Use induction on the cardinality of A to show that for all sets A of finite cardinality, $|\mathcal{P}(A)| = 2^{|A|}$. (Recall that $\mathcal{P}(A)$ denotes the power set of A .) As always in inductive proofs for CS378, clearly state the claim to be proved, prove the relevant base case, state the relevant inductive hypothesis, and prove the relevant inductive case.

4. Often in computer science and mathematics, we are given two different definitions of a structure and we need to determine if the two definitions are equivalent. In other words, we need to determine if the two definitions define the same thing. For example, consider the following two definitions of the set of strings of balanced parentheses:

Definition 1 The strings of balanced parentheses is the set of strings w over alphabet $\{ (,) \}$ such that

- (a) String w has an equal number of ('s and) 's, and
- (b) Any prefix of w has at least as many ('s as) 's.

(Note: A prefix of string w is a substring of w beginning at the first symbol in the string. Thus, if $w = (())$ then $(, ((, (()$, and $((())$ are among the prefixes of w .)

Definition 2

- (a) The empty string ϵ is balanced.
- (b) If x is a balanced string then (x) is also balanced.
- (c) If x and y are balanced strings then xy is also balanced.
- (d) Nothing else is a balanced string.

Your task is to prove that these two definitions are equivalent.

- *Hint:* To do this, let A denote the set of strings defined by Definition 1 and let B denote the set of strings defined by Definition 2. We would like to show that $A = B$. To do this, we can show that $A \subseteq B$ and that $B \subseteq A$. Each of these will require a separate proof by induction and each inductive proof takes some work. Be very precise and rigorous in your proof. It will take some work and careful reasoning. In particular, an easy pitfall is to accidentally use some intuitive understanding of balanced parentheses instead of only appealing to the definitions and using rigorous proof.

As always for inductive proofs, please remember to clearly state what is being proved, prove the relevant base case, state the relevant inductive hypothesis, and prove the relevant inductive case.

- *Hint:* If standard mathematical induction doesn't work, try strong induction for inductive proofs! (See pg. 23 your Sipser textbook.) Also, recall that structural induction can be useful for proving that a property holds for all elements of a recursively defined set.
- *Hint:* When proving $A \subseteq B$, consider non-empty string w in A . What if w had some proper, non-empty prefix x that also had equal numbers of ('s and) 's?

5. Give DFA's for each of the following languages over the alphabet $\{0, 1\}$:

- (a) $\{w \mid w \text{ has length at least 3 and an even number of 1's}\}$.
- (b) $\{w \mid w \text{ does not contain the pattern 00 but does contain at least one occurrence of the pattern 11}\}$.
- (c) $\{w \mid \text{Every 3 consecutive symbols in } w \text{ contains at least one 1}\}$.

For these exercises, please give both a state-transition machine diagram and an English explanation for each DFA. Please be sure your answers contain enough information to unambiguously determine the full 5-tuple for each DFA! (One good way to do that is to explicitly give the 5-tuple, with δ defined by the diagram.)