

Algorithm Design and Analysis
CS 376, Colby College
An Example Induction Proof¹

These notes include an example induction proof, a proof of the theorem

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

for all natural numbers $n \geq 1$.

Part 1 State what is being proved:

To prove: For all $n \in \mathbb{N}$, $n \geq 1$, $\sum_{i=1}^n i = \frac{n(n+1)}{2}$. Proof by induction on n .

To be complete in an induction proof, it's a good idea to say explicitly that the proof is by induction. You can add that to the “To prove” part of your proof, or you could add a separate line to the beginning of your proof, so it might look like this before continuing with the next steps:

Claim: For all $n \in \mathbb{N}$, $n \geq 1$, $\sum_{i=1}^n i = \frac{n(n+1)}{2}$.

Proof: Proof by induction on n .

Part 2 State and prove the *base case*:

Base case: The base case is $n = 1$. To prove that $\sum_{i=1}^1 i = \frac{1 \cdot (1+1)}{2}$, I start with the left side and show it is equal to the right:

$$\begin{aligned} \sum_{i=1}^1 i &= 1 && \text{by definition of summation} \\ &= \frac{1 \cdot (1+1)}{2} && \text{by arithmetic } \checkmark \end{aligned}$$

Part 3 State the *inductive hypothesis*:

Inductive Hypothesis: Assume that for arbitrarily chosen $k \geq 1$, $\sum_{i=1}^k i = \frac{k(k+1)}{2}$.

This is the assumption that what we're trying to prove is true for the case $n = k$; this assumption is then used to prove the inductive case (Part 4) where $n = (k + 1)$.

¹With thanks to Prof. Luke Hunsberger (Vassar College Computer Science), who used a similar handout in his courses.

Part 4 State and prove the *inductive case*:

Inductive Case: Using the inductive hypothesis, prove that $\sum_{i=1}^{k+1} i = \frac{(k+1)((k+1)+1)}{2}$. To do this, I start with the left side and show it is equal to the right:

$$\begin{aligned}
 \sum_{i=1}^{k+1} i &= \sum_{i=1}^k i + (k+1) && \text{by definition of summation} \\
 &= \frac{k(k+1)}{2} + (k+1) && \text{by inductive hypothesis} \\
 &= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} && \text{introduce a common denominator} \\
 &= \frac{k+1}{2} \cdot (k+1+1) && \text{by factoring out } \frac{k+1}{2} \\
 & && \text{and writing } k+2 \text{ as } k+1+1 \\
 &= \frac{(k+1)((k+1)+1)}{2} && \text{arithmetic } \checkmark
 \end{aligned}$$

That concludes the proof by induction!

Note that together, Part 3 and Part 4 show the following:

If what we're trying to prove is true for $n = k$, then it is true for $n = (k+1)$.

That is, we assume that what we're trying to prove is true for $n = k$, and using that assumption, we show it is true for $n = (k+1)$.

Following our domino analogy from lecture, these two steps together can be seen as saying that “all the dominoes are lined up right”—that is, if one falls, the next one falls, too. Putting that together with the base case (i.e., “the first domino falls”), this completes the “domino” version of the proof.

The next page contains the proof, without any instructional notes, just to have it presented complete and in one place.

To prove: For all $n \in \mathbb{N}$, $n \geq 1$, $\sum_{i=1}^n i = \frac{n(n+1)}{2}$. Proof by induction on n .

Base case: The base case is $n = 1$. To prove that $\sum_{i=1}^1 i = \frac{1 \cdot (1+1)}{2}$, I start with the left side and show it is equal to the right:

$$\begin{aligned} \sum_{i=1}^1 i &= 1 && \text{by definition of summation} \\ &= \frac{1 \cdot (1+1)}{2} && \text{by arithmetic } \checkmark \end{aligned}$$

Inductive Hypothesis: Assume that for arbitrarily chosen $k \geq 1$, $\sum_{i=1}^k i = \frac{k(k+1)}{2}$.

Inductive Case: Using the inductive hypothesis, prove that $\sum_{i=1}^{k+1} i = \frac{(k+1)((k+1)+1)}{2}$. To do this, I start with the left side and show it is equal to the right:

$$\begin{aligned} \sum_{i=1}^{k+1} i &= \sum_{i=1}^k i + (k+1) && \text{by definition of summation} \\ &= \frac{k(k+1)}{2} + (k+1) && \text{by inductive hypothesis} \\ &= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} && \text{introduce a common denominator} \\ &= \frac{k+1}{2} \cdot (k+1+1) && \text{by factoring out } \frac{k+1}{2} \\ & && \text{and writing } k+2 \text{ as } k+1+1 \\ &= \frac{(k+1)((k+1)+1)}{2} && \text{arithmetic } \checkmark \end{aligned}$$