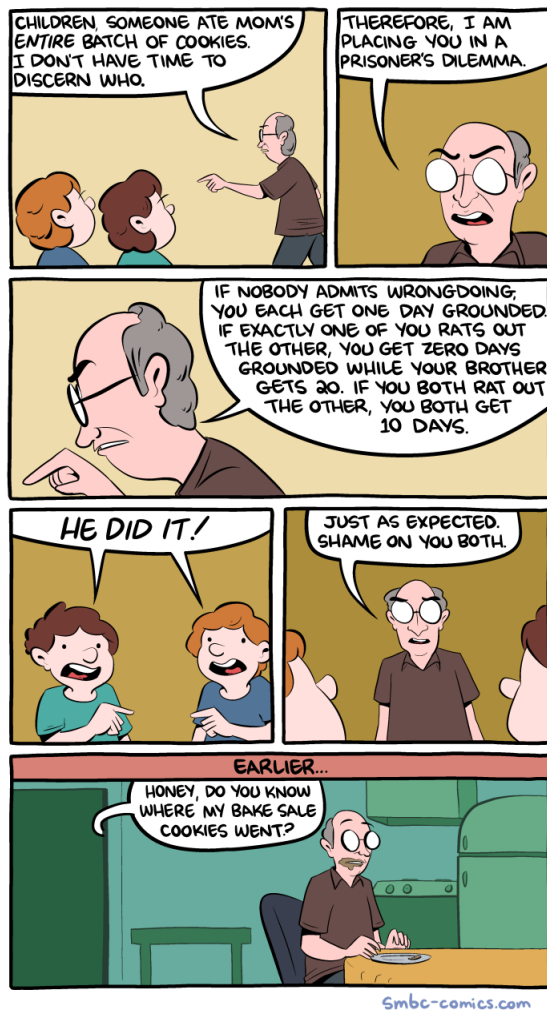


1 Prisoner's Dilemma



Probably the most classic and well known problem from Game Theory, the prisoner's dilemma has a long history of study (see its [wikipedia article](#) for a decent read). We can model this problem by a *payoff matrix* (or equivalently, by just multiplying all values by -1, a *cost matrix*), which specifies the payoffs to each player. As an example, consider the following cost matrix of values depicting the costs of the two children according to the comic:

$P_1 \backslash P_2$	abstain	tattle
abstain	1\1	20\0
tattle	0\20	10\10

By looking over the matrix of values, we can observe that without knowing how the other player will play, each player wants to blame the other. This is called a *dominant strategy*.

Definition 1.1. Player P 's **best response** to every other players' specification of strategies is

a strategy that maximizes the expected payoff to P assuming everyone else plays the specified strategies.

Definition 1.2. A **Dominant Strategy** is one in which by following the player is using (in at least one case strictly) a best response regardless of other players' chosen strategies.

As dominant strategies satisfy an extremely limiting condition and are not always present, we often look for a somewhat more relaxed condition called a *Nash Equilibrium*.

Definition 1.3. A **Nash Equilibrium** is a specification of strategies for each player such that no player wants to unilaterally deviate from their specified strategy.

2 Pollution Game

Let's imagine the USA wherein each state is drafting guidelines to control tax pollution in the United States according to the following rules:

Rule 1: States can impose some rules costing \$3 to enforces pollution control, which is a cost pushed to all living in state

Rule 2: For every state that pollutes, everyone in the nation pays a unit of cost

What naturally happens? Suppose everyone begins by paying the \$3 to control pollution. The state of Florida then starts thinking to themselves "Hey, what am I paying \$3 for? If I just get rid of my guidelines, I pay \$1 because I'm the only state polluting. I'm sure no one else will notice..."

So now Florida starts polluting and they are paying \$3 while everyone else is now paying \$4. Then Maine looks over and says "Well I want to pay less too!", so they stop enforcing guidelines, meaning they pay the \$1 they already were paying on account of Florida and an additional \$1 for their own pollution. The pattern begins to emerge...

Claim: the dominant strategy for each state is to not pay, regardless of how many other states are paying.

Imagine k states are currently paying the \$3. The cost of total pollution equals $n - k$. The total cost each state pays is $n - k + 3$ if they pay, vs the $n - k + 1$ cost if they do not pay. Since $n - k + 1 < n - k + 3$ for all k , it follows that every state's dominant strategy is to not enforce pollution guidelines and that the lack of any guidelines for any states is the Nash Equilibrium of this game.

3 Game of Chicken

Imagine two players playing the classic game of Chicken, where they drive straight at each other until one person "chickens out" by swerving out of the way or until they collide. We can imagine a representation of the payoffs of this game via the following payoff matrix:

$P_1 \backslash P_2$	Swerve	Straight
Swerve	0\0	-1\1
Straight	1\ -1	-100\ -100

Note that the top left box is not a Nash equilibrium because P_1 (and P_2) would prefer to have gone straight. The strategies corresponding to the bottom right box also is not a Nash equilibrium as P_1 (and P_2) would have preferred to swerve. Both the remaining two boxes, however, are Nash equilibrium's because if the swerver chooses to go straight then each players loses far worse than if the swerver would swerve, whilst the straight player doesn't have any impetus to go straight.

4 Tragedy of the Commons

In this problem, we imagine a network utilized by a collection of agents $\{1, 2, \dots, n\}$. Each agent i would be allowed to choose some $x_i \in [0, 1]$ proportion of the network bandwidth to occupy. However, the speed $1 - \sum_j x_j$ of the network degrades as it becomes more utilized. As a result, the net value of the allotted bandwidth agent i selects is $x_i(1 - \sum_j x_j)$.

For the sake of this example, let's imagine that each player selects their usage assuming everyone else maintains their current utilization. To do so, we assume that each player wants to maximize their value of their selection. We can maximize the value of a function by taking the derivative! The derivative of $x(1 - x)$ is $1 - 2x$, which yields $x = \frac{1}{2}$. Therefore the total value that player one receives is $\frac{1}{2}(1 - \frac{1}{2}) = \frac{1}{4}$.

When the second player makes their selection, we can do the same thing! This time they want to maximize $x * (1 - \frac{1}{2} - x) = x(\frac{1}{2} - x)$, whose derivative is $\frac{1}{2} - 2x$ which has a root at $x = \frac{1}{4}$.

But now P_1 would want to reconsider their utilization, as they want to maximize $x(1 - \frac{1}{4} - x)$ rather than the original $x(1 - x)$, this player would yield a derivative of $\frac{3}{4} - 2x$, giving $x = \frac{3}{8}$. Clearly, whenever a new player arrives or a player changes their selected utilization, each player would want to reevaluate their strategies. Hence there is no dominant strategy.

Driving Question What is the equilibrium of this game?

To do this, let's try and understand player i 's calculations. Letting t denote the sum of all other players' chosen utilizations, they want to maximize the following function: $x_i * (1 - t - x_i)$, the derivative with respect to x_i of which is $1 - t - 2x_i$, the root of which is found at $x_i = \frac{1-t}{2}$. Hence, by knowing the sum of all other players' chosen utilizations, player x_i can maximize their valuation by choosing $x_i = \frac{1-t}{2}$.

Let's assume that all players are symmetric, so that $x_i = x$ for all i for some value x . Since t represents that sum of all of the other players, then under this symmetric assumption we have $t = (n - 1)x$, so now we just solve for x ! By solving for x we have $x = \frac{1-t}{2} = \frac{1-(n-1)x}{2} = \frac{1}{n+1}$.

Hence, if everyone is choosing $\frac{1}{n+1}$, then each player is choosing their best response subject to everyone else's chosen strategies, and as a result we have a Nash equilibrium!

But what value is everyone getting? As each player is choosing $x_i = \frac{1}{n+1}$, each player has net value $\frac{1}{n+1}(1 - \frac{n}{n+1}) = \frac{1}{(n+1)^2}$.

What if, instead, everyone took $x_i = \frac{1}{2n}$? Each player's value would then be $\frac{1}{2n} * (1 - \frac{1}{2}) = \frac{1}{4n}$, where $\frac{1}{4n} > \frac{1}{(n+1)^2}$. However, clearly this is no players' best response to the other players' specified utilizations, and as a result this solution (despite giving the maximal total summed value over all players) is not stable.

5 Rock, Paper, Scissors

Games like rock, paper, scissors have some sort of clear "optimal" strategy that those of us in the room are able to identify, but it seems like our techniques today aren't able to really establish that strategy. The reason is that everything we've done today has been "deterministic", but there is clearly no deterministic Nash equilibrium for such games. Next time we'll discuss mixed strategies and mixed Nash equilibria, where we allow players to use randomness in their strategies.