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From the previous class, we have the following conditions:

1. Unanimity: If everyone agrees $A \geq B$ then the resultant ranking should have $A \geq B$.
2. Independence of Irrelevant Alternatives (IIA): Given two ranking specifications overall voters and candidates, where every voter's preference between A and B remains unchanged, the resulting ranking between the two inputs is unchanged between A and B

Theorem 11.1. *Arrow's Impossibility Thm '51: If the number of candidates is at least 3, then any social welfare function that satisfies Unanimity and independence of irrelevant alternatives is a dictatorship.*

By Arrow's Theorem above, we know there are no "very good" social welfare conditions.

Notation:

We'll use $<$ to denote an overall ranking scheme

We'll use $<_i$ to denote the i th voter's ranking

We'll use $<_F$ to denote the Social Welfare Functions(SWF) F 's resultant ranking

Given two ranking schemes, $<'$ and $<^*$ over all voters, where the voters who prefer A over B in the first ranking are exactly those who prefer C to D in the second, then $A >'_F B$ exactly when $C >^*_F D$

1.1 Proof:

Goal: take a social welfare condition where Unanimity and Independence of Irrelevant Alternatives hold, then prove that it is a dictatorship.

Here, results from the two votes are shown, but only focus on A and B first day and C and D second day

$<' : <^* : \text{Establish that if } V_F A >' B, C >^* D.$

v_1	v_2	v_3	$\dots$	v_n
A	B			A
B	A	B		
		A		B

v_1	v_2	v_3	$\dots$	v_n
C	D	D		
D		C		C
	C			D

Imagine a new ranking scheme, $<''$: take C and put it directly above A, take D and put it directly below B. $<''$: Results:

v_1	v_2	v_3	$\dots$	v_n
C	B			C
A	D	B		A
B	C	D		
D	A	C		B
		A		D

- By unanimity, $C > A$ and $D > A$ based on how we constructed
- By IIA, no voter has changed their internal preference for A and B, so whatever was before should still hold. Therefore $A > B$.
- exactly those who prefer $C > D$ in middle, prefer $C > D$ in final, since they match with $A > B$

Last time, we ended class with a simulation where we found the dictator.

Imagine a voting scheme $<_i$ where the first i voters prefer $A > B$ and the rest do not:

In $<^0$: $A <_F^0 B$

In $<^n$: $A >_F^n B$

(We get both by unanimity)

Let i^* be the vote where on the vote prior, $B >_F^{i^*-1} A$, but $A >_F^{i^*} B$

Claim: i^* voter is the dictator

(Note: there can be no ties in the output of social welfare functions.)

	V_1	V_2	V_3	$\dots$	V_{i^*-1}	V_{i^*}	V_{i^*+1}	$\dots$	V_n	F
$<^{i^*-1}$	A B	A B	A B		A B	B A	B A		B A	B A
$<^{i^*}$	A B	A B	A B		A B	A B	B A		B A	A B

We'll show if $D <_{i^*} C$ the $D <_F C$.

	V ₁	V ₂	V ₃	...	V _{i*-1}	V _{i*}	V _{i*+1}	...	V _n	F
$<^{j*-1}$	A B	A B	A B		A B	B A	B A		B A	B A
$<^{i*}$	A B	A B	A B		A B	A B	B A		B A	A B
$<$	C D	D C	D C		D C	C D	D C		D C	

Pick any other candidate E (that's why we need at least 3 candidates), construct $<_E$ For voters after i^* : E is the first choice For i^* , E is in the middle of C and D For voters after i^* : E is the last choice C and D don't change.

	V ₁	V ₂	V ₃	...	V _{i*-1}	V _{i*}	V _{i*+1}	...	V _n	F
$<^{j*-1}$	A B	A B	A B		A B	B A	B A		B A	B A
$<^{i*}$	A B	A B	A B		A B	A B	B A		B A	A B
$<$	C D	D C	D C		D C	C D	D C		D C	
$<^E$	E C D	E D C	E D C		E D C	C E D	D C E		D C E	

Look at voter's preferences between E and C: (E,C) (A,B) in $<^{i*-1}$ from the above proof, we know that two votes where those who vote for candidates the same way have the same results: The results have to be the same: here $C >_F^E E$ (E,D) (A,B): in $<^{i*}$: $E >_F^E D$ Together that gives $C >_F^E D$ Then by IIA, $C > D$ The resultant function exactly matches that voter for every function $\rightarrow v_{i*}$ is the dictator IIA gives symmetry between the elections

Note about 2 candidates: plurality works: fits both unanimity and IIA bc no third candidate to

“worry about dragging around”

Finally, we have the following from Arrow’s Thm:

Theorem (Corollary of Arrows thm): An election system is *incentive – compatible* if all players are encouraged to vote honestly.

Arrows thm has implications for choice functions despite being about social welfare functions: specifically, (*Gibbard – Satterthwaite* thm ‘73): Any social choice function where at least 3 candidates can win satisfies incentive compatibility must be a dictatorship.

Proof (handwavy): If you make a ranking by taking the top choice and putting it to the bottom repeatedly, this fulfills both unanimity and IIA.