

Introduction to Theory of Computation  
CS 378, Colby College  
Prof. Eric Aaron  
An Example Structural Induction Proof

These notes include a skeleton framework for an example *structural induction* proof, a proof that all propositional logic expressions (*PLEs*) contain an even number of parentheses.

Recall that structural induction is a method for proving statements about recursively defined sets. To show that a property  $\mathcal{P}$  holds for all elements of a recursively defined set:

**Base Case(s)** Show that  $\mathcal{P}$  holds for every element in the basis for the recursive definition.

**Inductive Case(s)** Show that every constructor in the definition preserves property  $\mathcal{P}$ .

For example, consider our recursive definition of propositional logic expressions:

**Base** Given an initial set  $A$  of propositional letters (e.g.,  $p, q, r, \dots$ ), all elements of  $A$  are *PLEs*.

**Induction** If  $P, Q$  are *PLEs*, then the following are also *PLEs* (note that the parentheses are part of the expressions):

- $(\neg P)$
- $(P \wedge Q)$
- $(P \vee Q)$
- $(P \rightarrow Q)$
- $(P \leftrightarrow Q)$

Note that the capital letters  $P, Q$  are not propositional letters (which are lowercase)—they are variables that stand for propositional logic expressions. (For example,  $P$  could stand for  $((p \vee q) \wedge r)$ , so  $(\neg P)$  would be  $(\neg((p \vee q) \wedge r))$ , etc. In general,  $P, Q$  can stand for any propositional logic expressions.)

Below is the framework of a proof that a property  $\mathcal{P}$  applies to all *PLEs*, where property  $\mathcal{P}(P)$  is “*PLE*  $P$  contains an even number of parentheses.”

Consider the claim:

**To prove:** All propositional logic expressions contain an even number of parentheses. (We consider 0 to be an even number.)

We prove the Claim holds for all PLEs by structural induction—first, we show it holds for all propositional letters (the base case), and then we show that each of the five constructors of PLEs preserves the truth of it; thus, because all PLEs are either propositional letters or the result of applying a constructor to existing PLEs, we have shown it holds for all PLEs. (That’s structural induction!)

**Proof.** We start with the **Base Case**: For every propositional letter  $p$ ,  $p$  has 0 parentheses, which we consider to be an even number.

For the **Inductive Case**, we have five sub-proofs. Because all five constructors can be seen as building new PLEs from at most two existing PLEs, we will use  $P$  and  $Q$  to stand for arbitrarily chosen PLEs in the cases that follow. (**What should the inductive hypothesis be?**) The cases correspond to the constructors:

1. **Constructor**  $(\neg P)$  For this, the I.H. is that  $P$  has an even number of parentheses, and we show that under that hypothesis,  $(\neg P)$  does, as well. Let  $n$  stand for the number of parentheses in  $P$ . Then ... (how would you show that  $(\neg P)$  also has an even number of parentheses?)
2. **Constructor**  $(P \wedge Q)$  For this, the I.H. is that each of  $P$  and  $Q$  has an even number of parentheses, and we show that under that hypothesis,  $(P \wedge Q)$  does, as well. For notation, let  $n_P$  stand for the number of parentheses in  $P$  and  $n_Q$  stand for the number of parentheses in  $Q$ . Then ... (how would you show that  $(P \wedge Q)$  also has an even number of parentheses?)
3. **Constructor**  $(P \vee Q)$  ... (this is very similar to case (b), although a full proof would include it in detail).
4. **Constructor**  $(P \rightarrow Q)$  ... (this is very similar to case (b), although a full proof would include it in detail).
5. **Constructor**  $(P \leftrightarrow Q)$  ... (this is very similar to case (b), although a full proof would include it in detail).

Having now proved that property  $\mathcal{P}$  holds for every base case (the propositional letters) and that each constructor preserves  $\mathcal{P}$ —that is, if  $\mathcal{P}$  is true for the things on which the constructors operate, then  $\mathcal{P}$  is true for the result of the constructor—this above proof by structural induction shows that  $\mathcal{P}$  holds for every PLE.